BOUNDING SUP-NORMS OF CUSP FORMS OF LARGE LEVEL

VALENTIN BLOMER AND ROMAN HOLOWINSKY

ABSTRACT. Let f be an L^2-normalized weight zero Hecke-Maaß cusp form of square-free level N, character χ and Laplacian eigenvalue $\lambda \geqslant 1/4$. It is shown that $\|f\|_\infty \ll_\lambda N^{-1/37}$, from which the hybrid bound $\|f\|_\infty \ll \lambda^{1/4}(N\lambda)^{-\delta}$ (for some $\delta > 0$) is derived. The first bound holds also for $f = y^{k/2}F$ where F is a holomorphic cusp form of weight k with the implied constant now depending on k.

1. INTRODUCTION

If f is a Maaß cusp form on a (compactified) modular curve $X_0(N) = \Gamma_0(N)\backslash\mathbb{H}^*$, normalized with respect to the inner product

$$\langle f_1, f_2 \rangle = \int_{X_0(N)} f_1(z)\bar{f}_2(z)\frac{dxdy}{y^2}, \tag{1}$$

it is interesting and often useful to derive pointwise bounds for f. Such bounds play an important role for instance in connection with subconvexity of L-functions [HM, Proposition 4] and in connection with the mass equidistribution conjecture [Ru, Appendix]. In the latter example, bounds for the $\Gamma_0(N)$-invariant function $f(z) = y^{k/2}|F(z)|$ are obtained for a holomorphic cusp form $F \in \mathcal{S}_k(N, \chi)$ of weight k.

The problem of bounding cusp forms can be seen in the general framework of comparing two norms on a vector space. Of course, in the case of the finite-dimensional vector space $\mathcal{S}_k(N, \chi)$, the L^2-norm and the L^∞-norm must be equivalent, but interestingly, the finite-dimensionality of $\mathcal{S}_k(N, \chi)$ follows conversely from the equivalence of these two norms [Fr, Theorem 6.1]. Another perspective comes from bounding periods of automorphic forms (see e.g. [MV, Sa1]), and hence our work is at least implicitly related to the subconvexity problem for automorphic L-functions. We will come back to this point later. In our case, of course, the "period" is somewhat degenerate, namely a single point.

In the case $k = 2$, the average

$$\sup_{z \in \mathbb{H}} \sum_{F \in \mathcal{B}_2(N,1)} y^2|F(z)|^2$$

over an orthonormal basis $\mathcal{B}_2(N, 1)$ of $\mathcal{S}_2(N, 1)$ can be interpreted as the ratio of the Arakelov metric (coming from embedding $X_0(N)$ into its Jacobian) and the hyperbolic metric (coming from the Petersson inner product) on $X_0(N)$. Following [AU, MU], Jorgenson and Kramer [JK1] derived in a more general context the best-possible bound

$$\sup_{z \in \mathbb{H}} \sum_{F \in \mathcal{B}_2(N,1)} y^2|F(z)|^2 = O(1) \tag{2}$$

as $N \to \infty$. Their line of investigation is continued in the important work [JK2].

In order to treat the holomorphic and non-holomorphic case simultaneously, we define the archimedean parameter of a cusp form f as

$$(3) \qquad t = t_f = \begin{cases} \frac{(k-1)}{2}, & f = y^{k/2}F \text{ for a holomorphic form } F \text{ of weight } k, \\ \sqrt{\lambda_f - \frac{1}{4}}, & f \text{ a Maaß form satisfying } \Delta f = \lambda_f f \text{ for } \Delta = -y^2(\partial_x^2 + \partial_y^2). \end{cases}$$

The choice of the square root is irrelevant. We call a Maaß form f non-exceptional if $\lambda_f \geqslant 1/4$, i.e. $t_f \in \mathbb{R}$. Pointwise bounds for L^2-normalized cusp forms can be in terms of the archimedean parameter t, the level N, or both. The first breakthrough in the t-aspect was achieved by Iwaniec and Sarnak [IS] who proved

$$(4) \qquad \|f\|_\infty \ll_\varepsilon (1 + |t_f|)^{5/12+\varepsilon} \|f\|_2.$$

for a weight zero Hecke-Maaß cusp form of level 1, and any $\varepsilon > 0$. The "trivial" bound in this context is $(1 + |t_f|)^{1/2}$. One might conjecture that $\|f\|_\infty \ll_\varepsilon (1 + |t_f|)^\varepsilon \|f\|_2$ holds for all $\varepsilon > 0$, but if at all, this can only hold for compact Riemann surfaces: high in the cusp one has [Iw1, p. 178]

$$f\left(\frac{1}{4} + \frac{it_f}{2\pi}\right) = (1 + |t_f|)^{1/6+o(1)} \|f\|_2.$$

For holomorphic forms the situation is easier, because here the upper bound that comes from the Fourier expansion is essentially sharp. More precisely, it was recently shown in [Xia] that

$$k^{1/4-\varepsilon}\|f\|_2 \ll_\varepsilon \|f\|_\infty \ll_\varepsilon k^{1/4+\varepsilon}\|f\|_2$$

for $f = y^{k/2}F$ where F is a holomorphic Hecke cusp form of level 1 and weight k. For a sup-norm bound in a somewhat different context see also [Ku].

In the level aspect, nothing beyond the average bound (2) is known. For an individual Hecke cusp form this recovers only the trivial bound that comes from the Fourier expansion, see Lemma 7 below. On the other hand, since the volume of $X_0(N)$ is about N, one might (optimistically) conjecture

$$(5) \qquad \|f\|_\infty \ll_\varepsilon N^{-1/2+\varepsilon} \|f\|_2.$$

for fixed archimedean parameter t and a Hecke eigenform f. If this was true, it is not only best possible, but would also be very strong in the sense that it implied as a special case the Lindelöf hypothesis for automorphic L-functions in the level aspect, since $f(i/\sqrt{N}) \approx L(1/2, f)N^{-1/2}\|f\|_2$. This shows, on the other hand, that in order to derive some subconvex bound for $L(1/2, f)$ in the level aspect by this method, one would need already a relatively strong pointwise bound $\|f\|_\infty \ll N^{-1/4-\delta}\|f\|_2$ for some $\delta > 0$. In a different direction, (5) would imply the best possible bounds

$$(6) \qquad \|f\|_p \ll_{\varepsilon,p} N^{\frac{1}{p}-\frac{1}{2}+\varepsilon}, \quad p \geqslant 1.$$

for the L^p-norm of an L^2-normalized Hecke cusp form f.

Certainly (5) is true if f is an oldform that comes from a level 1 form. It should be observed, however, that the requirement that f is a Hecke eigenform cannot be dropped, as can be seen, for example, by looking at (holomorphic) Poincaré series

$$P_m(z) := \sum_{n=1}^\infty \left(\delta_{nm} + 2\pi i^{-k} \sum_{N|c} \frac{1}{c} S_\chi(m, n; c) J_{k-1}\left(\frac{4\pi\sqrt{nm}}{c}\right) \right) \left(\frac{n}{m}\right)^{(k-1)/2} e(nz) \in \mathcal{S}_k(N, \chi).$$

If we regard $m \geqslant 1$ and $k \geqslant 4$ as fixed and N large, then $P_m(i) \asymp 1$, while it is easily seen that $\|y^{k/2}P_m\|_2 \asymp 1$, cf. e.g. [Iw2, (3.24)].

The aim of this article is to establish the first non-trivial bound of the L^∞-norm of an individual L^2-normalized holomorphic or non-holomorphic cuspidal Hecke eigenform of large square-free level N. For notational simplicity let us write

$$t^* := 1 + |t_f|$$

with t_f as in (3).

Theorem 1. *Let $N \geqslant 1$ be a square-free integer, $k \geqslant 2$ an even integer and χ an even Dirichlet character modulo N. Let f be an L^2-normalized non-exceptional weight zero Hecke-Maaß cusp form of level N and character χ, or let $f = y^{k/2}F$ where $F \in \mathcal{S}_k(N, \chi)$ is an L^2-normalized Hecke eigenform. Then*

$$\|f\|_\infty \ll (t^*)^{\frac{11}{2}} N^{-\frac{1}{37}}.$$

The implied constant is absolute.

As a first step towards (6), this implies

Corollary 1. *Under the same assumptions as in Theorem 1, one has*

$$\|f\|_p \ll_t N^{-\frac{p-2}{37p}}$$

whenever $p \geqslant 2$. In particular, $\|f\|_4 \ll_t N^{-1/74}$.

As far as we know, no non-trivial bound for any L^p-norm, $p > 2$, in the N-aspect has ever been obtained. In the eigenvalue aspect, a very strong bound for the L^4-norm of level 1 Hecke-Maaß cusp forms has been proved by Sarnak and Watson, see [Sa, p. 460] for more details and [Sp] for the Eisenstein case.

The method of the proof of Theorem 1 works also for holomorphic and non-holomorphic cusp forms of any weight with $(t^*)^{11/2}$ replaced by $(|\lambda| + |k|)^A$. While we tried to be efficient in the N-aspect, we made little attempt to optimize the t^*-aspect. It is instructive to see that a nontrivial sup-norm bound is related to solving a very concrete diophantine problem (cf. Sections 7 and 8), and the focus here lies on a transparent treatment of the underlying number theoretical features rather than a tedious game with Bessel functions and oscillatory integrals. Our aim was, however, to provide *some* explicit t^*-exponent in Theorem 1 with the following application in mind. Analyzing the proof of (4), it is not hard to see that it is essentially uniform in N: we will prove in Proposition 1 in Section 10 that

(7) $$\|f\|_\infty \ll_\varepsilon N^\varepsilon (t^*)^{5/12+\varepsilon}$$

for an L^2-normalized Hecke-Maaß cusp form of square-free level N. Taking this for granted for the moment, one obtains the following hybrid bound:

Theorem 2. *If f is a Maaß form as in Theorem 1, one has*

$$\|f\|_\infty \ll (t^*)^{1/2}(t^*N)^{-\frac{1}{2300}}$$

with an absolute implied constant.

Indeed, one has $\min((t^*)^5 N^{-\frac{1}{37}}, (t^*)^{-\frac{1}{12}}) \leqslant ((t^*)^5 N^{-\frac{1}{37}})^{\frac{37}{2269}}((t^*)^{-\frac{1}{12}})^{\frac{2232}{2269}} \leqslant (t^*N)^{-\frac{1}{2269}}$.

We briefly discuss the plan of attack. By Atkin-Lehner theory we can restrict z to a Siegel set with $y \gg 1/N$ (here we use the fact that N is square-free). Thus we will have to detect cancellation in sums of the type $\sum_{n \sim N} \lambda_f(n)e(nx)$ where we have no control on x, and little control on the Hecke eigenvalues $\lambda_f(n)$, since the level of f is large. Generalizing [MU], we consider an amplified second moment. It is clear a priori that this setup alone cannot guarantee success, since in general the error term of the un-amplified second moment has about the same size as the main term. However, a careful analysis shows that for x off rational points with small denominator, this method does give a saving. If x is well-approximable, a Voronoi-type argument gives a non-trivial result. The dichotomy of well and badly-approximable x seems to be new in this context and does not appear

in the work of Iwaniec-Sarnak [IS]. It may remind of the circle method, and we shall refer to the set of well-approximable x as the major arcs.

The paper is organized as follows: In Section 2-4 we recall some well-known material that we will need for the proof of Theorem 1. Section 5 gives a first bound for $f(x + iy)$ that will turn out to be useful for well-approximable x. Sections 6-9 implement the amplification method and give a second bound that is strong for badly-approximable x. We finish the proof of Theorem 1 in Section 9, and we prove (7) in Section 10 which implies immediately Theorem 2. Finally we remind the reader of the usual convention that the value of ε may change during a calculation.

Acknowledgement: The authors thank Antal Balog, Matthew Young, Gergely Harcos, Philippe Michel, Jean Bourgain, Peter Sarnak and the referees for important and useful comments on various aspects of this work.

2. Automorphic Forms and Fourier expansion

Let $N \geqslant 1$ be a square-free integer, χ an even character to modulus N; let $k \geqslant 2$ be an even integer. We denote by $\mathcal{L}^2(N, \chi)$ and $\mathcal{L}_0^2(N, \chi) \subset \mathcal{L}^2(N, \chi)$, respectively, the spaces of weight zero Maaß forms, and of weight zero Maaß cusp forms with respect to the congruence subgroup

$$\Gamma_0(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in SL_2(\mathbb{Z}) : N \mid c \right\}$$

and with nebentypus χ. In slight contrast to the usual notation, we shall write $\mathcal{S}_k(N, \chi)$ for the set of all functions $f(z) = y^{k/2} F(z)$ where F is a holomorphic cusp form of weight k. Both $\mathcal{L}_0^2(N, \chi)$ and $\mathcal{S}_k(N, \chi)$ are Hilbert spaces with respect to the inner product (1). We can choose bases $\mathcal{B}_k(N, \chi)$ and $\mathcal{B}(N, \chi)$ of $\mathcal{S}_k(N, \chi)$ and $\mathcal{L}_0^2(N, \chi)$ that consist of L^2-normalized eigenforms for all Hecke operators T_n with $(n, N) = 1$. For the proof of Theorem 1 we can assume that f is a newform. We define the archimedean parameter of a cusp form f as in (3) and write for notational simplicity

$$t^* := 1 + |t|$$

as above. It is known [KS] that $|\Im t| \leqslant \theta := \frac{7}{64}$. We write $\mathcal{B}(t, N, \chi) \subseteq \mathcal{B}(N, \chi)$ for the set of those $f \in \mathcal{B}(N, \chi)$ with archimedean parameter t. The orthogonal complement to $\mathcal{L}_0^2(N, \chi)$ in $\mathcal{L}^2(N, \chi)$ is the Eisenstein spectrum $\mathcal{E}(N, \chi)$ (plus possibly the space of constant functions if χ is trivial). The adelic reformulation of the theory of automorphic forms provides a natural spectral expansion of this space in which the basis of Eisenstein series is indexed by a set of parameters of the form

$$(8) \qquad \{(t, \chi_1, \chi_2, b) \mid t \in \mathbb{R}, \chi_1\chi_2 = \chi, \ b \in \mathcal{B}(\chi_1, \chi_2)\},$$

where $\mathcal{B}(\chi_1, \chi_2)$ is some finite (possibly empty) set[1]. We refer to [GJ] for the definition of these parameters as well as for the proof of the corresponding version of the spectral theorem. The reason why we introduce this particular basis of $\mathcal{E}(N, \chi)$ is that it consists of Hecke eigenforms for all Hecke operators T_n with $(n, N) = 1$ where the Hecke eigenvalue of the corresponding Eisenstein series $E_{t,\chi_1,\chi_2,b}$ is given by

$$\lambda_{t,\chi_1,\chi_2,b}(n) = \sum_{d_1 d_2 = n} \chi_1(d_1) d_1^{it} \chi_2(d_2) d_2^{-it}.$$

Of course, these Eisenstein series can be described in purely classical terms without recourse to adeles. If a cusp form f is an eigenform of some Hecke operator T_n, we denote the corresponding eigenvalue by $\lambda_f(n)$. A form f in $\mathcal{B}_k(N, \chi)$, $\mathcal{B}(N, \chi)$, or an Eisenstein series $f = E_{t,\chi_1,\chi_2,b}$ has a Fourier expansion

$$f(z) = \rho_f(0, y) + \sum_{n \neq 0} \rho_f(n) W_f(|n|y) e(nx)$$

---

[1]specifically, $\mathcal{B}(\chi_1, \chi_2)$ corresponds to an orthonormal basis in the space of an induced representation constructed out of the pair (χ_1, χ_2), but we do not need to be more precise.

where

$$(9) \qquad W_f(y) = \begin{cases} \Gamma(k)^{-1/2}(4\pi y)^{k/2}e^{-2\pi y}, & f \in \mathcal{B}_k(N,\chi), \\ y^{1/2}\cosh(\tfrac{1}{2}\pi t)K_{it}(2\pi y), & f \in \mathcal{B}(t,N,\chi) \text{ or } f = E_{t,\chi_1,\chi_2,b} \end{cases}$$

for $y > 0$,

$$(10) \qquad \rho(-n) = \eta\rho(n)$$

for $n > 0$ and $\eta \in \{-1,0,1\}$, and $\rho_f(0,y) = 0$ unless f is Eisenstein. The case $\eta = 0$ occurs for holomorphic forms. By our choice of (Hecke) bases for $\mathcal{S}_k(N,\chi), \mathcal{L}_0^2(N,\chi)$ and $\mathcal{E}(N,\chi)$ one has for any basis form f, any $n \geqslant 1$, and $(m,N) = 1$ the relation

$$(11) \qquad \lambda_f(m)\sqrt{n}\rho_f(n) = \sum_{d|(m,n)} \chi(d)\sqrt{\frac{mn}{d^2}}\rho_f\left(\frac{mn}{d^2}\right);$$

in particular,

$$(12) \qquad \lambda_f(m)\rho_f(1) = \sqrt{m}\rho_f(m)$$

for $(m,N) = 1$. Moreover, these relations hold for all m, n if f is a cuspidal newform. For such a cuspidal newform f we have the uniform bounds [DFI, Prop. 19.6]

$$(13) \qquad \sum_{n\leqslant X} |\lambda_f(n)|^2 \ll_\varepsilon X(XNt^*)^\varepsilon$$

for any $\varepsilon > 0, X \geqslant 1$,

$$(14) \qquad \frac{1}{N}(Nt^*)^{-\varepsilon} \ll_\varepsilon |\rho_f(1)|^2 \ll_\varepsilon \frac{1}{N}(Nt^*)^\varepsilon$$

(cf. [HM, (30)-(31)]) due to Iwaniec and Hoffstein-Lockhart, and

$$(15) \qquad |\lambda_f(n)| \ll_\varepsilon n^{\theta+\varepsilon}$$

for $\theta = 7/64$, see [KS]. Note that we have normalized the Whittaker function W_f in (9) differently than in [HM, p. 593]. It follows from Atkin-Lehner theory [AU, Section 2] that

$$(16) \qquad \sup_{z\in\mathbb{H}} |f(z)| = \sup_{y\geqslant\frac{\sqrt{3}}{2N}} |f(x+iy)|$$

for a cuspidal newform f. Here we use the fact that N is square-free.

3. SUMMATION FORMULAE

First we recall the Voronoi summation formulae [KMV, Appendix A3 and A4]. Let $q \geqslant 1$ be an integer, and write $N_1 := (N,q)$, $N_2 = N/N_1$. Since N is square-free, this implies $(N_1,N_2) = 1$. For a cusp form f and $y > 0$ define $\mathcal{J}^\pm = \mathcal{J}_f^\pm$ by

$$(17) \qquad \mathcal{J}^+(y) = 2\pi J_{k-1}(4\pi y), \quad \mathcal{J}^-(y) = 0$$

if f is holomorphic of weight k, and

$$(18) \qquad \mathcal{J}^+(y) = \frac{\pi}{\cosh(\pi t)}\left(Y_{2it}(4\pi y) + Y_{-2it}(4\pi y)\right), \quad \mathcal{J}^-(y) = 4\cosh(\pi t)K_{2it}(4\pi y)$$

if f is Maaß.

Lemma 1. *Let $(a,q) = 1$ and let g be a smooth function, compactly supported in $(0,\infty)$. Let f be a newform of level N. Then there exist two complex numbers $\eta_\pm$ of absolute value 1 (depending on a,q and f) and a newform f^* of the same level N and the same archimedean parameter such that*

$$(19) \qquad \sum_n \lambda_f(n)e\left(\frac{an}{q}\right)g(n) = \sum_\pm \frac{\eta_\pm}{q\sqrt{N_2}}\sum_n \lambda_{f^*}(n)e\left(\mp\frac{n\overline{aN_2}}{q}\right)\int_0^\infty g(\xi)\mathcal{J}^\pm\left(\frac{\sqrt{n\xi}}{q\sqrt{N_2}}\right)d\xi$$

where as usual the bar denotes the multiplicative inverse.

For $m, n, c \in \mathbb{N}$ let $S_\chi(m, n; c)$ denote the twisted Kloosterman sum

$$S_\chi(m, n; c) = \sum_{\substack{a\,(c) \\ (a,c)=1}} \bar{\chi}(a) e\left(\frac{m\bar{a} + na}{c}\right).$$

Let $\phi : [0, \infty) \to \mathbb{C}$ be a smooth function satisfying $\phi(0) = \phi'(0) = 0$, $\phi^{(j)}(y) \ll_\varepsilon (1 + y)^{-2-\varepsilon}$ for $0 \leqslant j \leqslant 3$. Let

(20)
$$\dot{\phi}(k) := i^k \int_0^\infty J_{k-1}(x)\phi(y)\frac{dy}{y},$$
$$\tilde{\phi}(t) := \frac{i}{2\sinh(\pi t)} \int_0^\infty \left(J_{2it}(y) - J_{-2it}(y)\right)\phi(y)\frac{dy}{y}$$

be Bessel transforms.

Lemma 2. *For positive integers m, n the trace formula of Bruggeman–Kuznetsov holds:*

$$\sum_c \frac{1}{Nc} S_\chi(m, n; Nc)\phi\left(\frac{4\pi\sqrt{mn}}{Nc}\right) = 4 \sum_{\substack{\kappa \geqslant 2\ even \\ f \in \mathcal{B}_\kappa(N,\chi)}} \dot{\phi}(k)\overline{\rho_f(m)}\rho_f(n)\sqrt{mn}$$

$$+ \sum_{f \in \mathcal{B}(N,\chi)} 4\pi\tilde{\phi}(t_f)\overline{\rho_f(m)}\rho_f(n)\sqrt{mn} + \sum_{\substack{\chi_1\chi_2=\chi \\ b\in\mathcal{B}(\chi_1,\chi_2)}} \int_{-\infty}^\infty \tilde{\phi}(t)\overline{\rho_{t,\chi_1,\chi_2,b}(m)}\rho_{t,\chi_1,\chi_2,b}(n)\sqrt{mn}\,dt.$$

For a proof see [Iw1, Theorem 9.8] and note the different normalization there[2] that is given by [Iw1, (1.26), (8.1), (8.2), (8.5), (8.6)]. The proof of this formula with the Eisenstein parameters (8) is identical. The holomorphic counterpart is Petersson's trace formula

Lemma 3. *We have*

$$\delta(m, n) + 2\pi i^{-k} \sum_c \frac{1}{Nc} S_\chi(m, n; Nc) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{Nc}\right) = \frac{4\pi\sqrt{mn}}{k-1} \sum_{f \in \mathcal{B}_k(N,\chi)} \overline{\rho_f(m)}\rho_f(n)$$

where $\delta(m, n) = 1$ if $m = n$ and $\delta(m, n) = 0$ otherwise.

This is proved in [Iw2, Theorem 3.6] for Fourier coefficients normalized as in [Iw2, p. 52].

Finally we need the following result. As usual, for $\alpha \in \mathbb{R}$ we write $\|\alpha\| := \min(|\alpha - n| : n \in \mathbb{Z})$.

Lemma 4. *Let $0 < T \leqslant Z$, $\alpha \in \mathbb{R}\setminus\mathbb{Z}$. Let Φ be a smooth function, compactly supported on $[Z/2, 2Z]$ such that $\Phi^{(j)} \ll_j T^{-j}$ for all $j \in \mathbb{N}_0$. Then*

$$\sum_m e(\alpha m)\Phi(m) \ll_j Z(T\|\alpha\|)^{-j}$$

for any $j \geqslant 2$.

Proof. This is a simple application of Poisson summation and repeated integration by parts.

4. Special functions

For future reference we record some properties of the functions W_f defined in (9) and $\mathcal{J}_f$ defined in (17) - (18). For $y > 0$, $\sigma > 0$, $s = \sigma + it \in \mathbb{C}$, $k \in \mathbb{N}_0$, $A \geqslant 0$ and $\varepsilon > 0$ we have the bounds

(21)
$$J_k(y) \ll \frac{1 + k}{1 + y^{1/2}},$$

---

[2]Note that equation [Iw1, (9.15)] should have the normalization factor $\frac{2}{\pi}$ instead of π, and in equation [Iw1, (B.49)] a factor 4 is missing.

$$(22) \qquad \frac{Y_s(y)}{\cosh(\pi t/2)} \ll_{\sigma,\varepsilon} \begin{cases} ((1+|t|)y^{-1})^{|\sigma|+\varepsilon}, & y \leqslant 1+|t|, \\ ((1+|t|)y^{-1})^{-\varepsilon}, & 1+|t| < y \leqslant 1+|s|^2, \\ y^{-1/2}, & y > 1+|s|^2, \end{cases}$$

(23)

$$\cosh(\pi t/2)K_s(y) \ll_{\varepsilon,\sigma} \begin{cases} ((1+|t|)y^{-1})^{|\sigma|+\varepsilon}, & y \leqslant 1+\frac{1}{2}\pi|t| \\ e^{-y+\pi|t|/2}y^{-1/2}, & y > 1+\frac{1}{2}\pi|t| \end{cases} \ll_A \left(\frac{1+|t|}{y}\right)^{|\sigma|+\varepsilon} \left(1+\frac{y}{1+|t|}\right)^{-A}.$$

The estimates (21) - (23) follow directly from [HM], Propositions 8 and 9. They are not best possible in all ranges, but sufficiently uniform for our present purposes. Moreover, we have

$$(24) \qquad J_r(y)' = \frac{1}{2}\left(J_{r-1}(y) - J_{r+1}(y)\right), \quad K_r(y)' = -\frac{1}{2}\left(K_{r-1}(y) + K_{r+1}(y)\right)$$

for arbitrary $r \in \mathbb{C}$. If $\alpha > 0$, $g \in \mathcal{C}_0^1((0,\infty))$, then it follows from the formulas $(y^r J_r(y))' = y^r J_{r-1}(y)$, $(y^r Y_r(y))' = y^r Y_{r-1}(y)$, $(y^r K_r(y))' = -y^r K_{r-1}(y)$ and one integration by parts that

$$(25) \qquad \int_0^\infty g(y) \begin{Bmatrix} J_r(\alpha\sqrt{y}) \\ Y_r(\alpha\sqrt{y}) \\ K_r(\alpha\sqrt{y}) \end{Bmatrix} dy = \begin{Bmatrix} + \\ + \\ - \end{Bmatrix} \frac{2}{\alpha} \int_0^\infty \left(g'(y)\sqrt{y} - \frac{r}{2}\frac{g(y)}{\sqrt{y}}\right) \begin{Bmatrix} J_{r+1}(\alpha\sqrt{y}) \\ Y_{r+1}(\alpha\sqrt{y}) \\ K_{r+1}(\alpha\sqrt{y}) \end{Bmatrix} dy.$$

Lemma 5. *Let f be a holomorphic cusp form or a non-exceptional Maaß cusp form (i.e. $\Im t_f = 0$). Then one has*

$$\frac{d^j}{dy^j}W_f(y) \ll_{j,\varepsilon,A} (t^*)^{1/2} \left(\frac{t^*}{y}\right)^{j+\varepsilon} \left(1+\frac{y}{t^*}\right)^{-A}$$

for any $j, A \geqslant 0$.

Proof. If f is Maaß, the lemma follows immediately from (9), the second equation in (24), and (23), and it conveniently simplifies further computations if we replace $y^{1/2}$ by $(t^*)^{1/2}$. If f is holomorphic, we observe that the function

$$W_f(y) = \Gamma(k)^{-1/2}(4\pi y)^{k/2}e^{-2\pi y}$$

has its maximum at $y_0 = k/(4\pi)$, and by Stirling's formula $W_f(y_0) \asymp k^{1/4}$. This readily confirms (a stronger version of) Lemma 5 for $j = 0$ and $y \leqslant 10k$, say. For $y > 10k$ we have again by Stirling's formula $W_f(y) \ll k^{1/4}\exp(-2\pi y + \frac{k}{2}\log\frac{4\pi ey}{k}) \leqslant k^{1/4}e^{-\pi y}$. For $j > 0$, we observe that $W_f^{(j)}(y) \ll_j (1+k/y)^j W_f(y)$ which completes the proof in the holomorphic case.

Lemma 6. *Let f be as in the preceding lemma. Let $1 \leqslant T \leqslant Z$, $\varepsilon > 0$ and let g be a smooth function, compactly supported on $[Z/2, 2Z]$ such that $g^{(j)} \ll_j T^{-j}$ for all $j \in \mathbb{N}_0$. Let $\alpha > 0$ and*

$$I := \int_0^\infty g(\xi)\mathcal{J}_f^\pm(\alpha\sqrt{\xi})d\xi.$$

Then

$$(26) \qquad I \ll_\varepsilon \frac{Z^{3/4}t^*}{\alpha^{1/2}}(t^*(\alpha+\alpha^{-1}))^\varepsilon$$

Moreover, if $\alpha\sqrt{Z/2} \geqslant 2t^*$, then

$$(27) \qquad I \ll_{\varepsilon,j} \left(\frac{1}{\alpha}\left(\frac{\sqrt{Z}}{T} + \frac{t^*}{\sqrt{Z}}\right)\right)^j \frac{Z^{3/4}t^*}{\alpha^{1/2}}(t^*(\alpha+\alpha^{-1}))^\varepsilon$$

for any $j \in \mathbb{N}_0$.

Proof. From (21) - (23) we infer $\mathcal{J}_f^\pm(\alpha\sqrt{\xi}) \ll_\varepsilon t^*(\alpha\sqrt{Z})^{-1/2}(t^*(\alpha+\alpha^{-1}))^\varepsilon$ for $\xi \in [Z/2, 2Z]$, and the first bound is immediate. For the proof of the second bound we apply j times the formula (25) and then estimate trivially using (21) - (23) as before.

Finally we specify a test function ϕ for the Kuznetsov formula (Lemma 2). For integers $2 \leqslant B < A$ of the same parity let us define

$$(28) \qquad \phi_{A,B}(x) := i^{B-A} J_A(x) x^{-B}.$$

Using [GR, 6.574.2] it is straightforward to verify that

$$(29) \qquad \begin{aligned} \dot{\phi}_{A,B}(k) &= \frac{B!}{2^{B+1}\pi} \prod_{j=0}^{B} \left\{ \left(\frac{(1-k)i}{2}\right)^2 + \left(\frac{A+B}{2} - j\right)^2 \right\}^{-1} \asymp_{A,B} \pm k^{-2B-2}, \\ \tilde{\phi}_{A,B}(t) &= \frac{B!}{2^{B+1}\pi} \prod_{j=0}^{B} \left\{ t^2 + \left(\frac{A+B}{2} - j\right)^2 \right\}^{-1} \asymp_{A,B} (1+|t|)^{-2B-2}. \end{aligned}$$

In particular,

$$(30) \qquad \begin{aligned} \dot{\phi}_{A,B}(k) &> 0 \quad \text{for} \quad 2 \leqslant k \leqslant A - B, \\ \tilde{\phi}_{A,B}(t) &> 0 \quad \text{for} \quad t \in \mathbb{R} \cup i[-\theta, \theta]. \end{aligned}$$

5. THE MAJOR ARCS

We keep the notation developed so far. For the proof of Theorem 1 we can assume that N is sufficiently large. It will be convenient to introduce a smooth partition of unity: Let $Z = 2^\nu$, $\nu \in \mathbb{N}_0$, run through powers of 2, and let G be a smooth nonnegative function G supported on $[1/2, 2]$ such that $\sum_Z G(x/Z) = 1$ for all $x \geqslant 1$. For a (non-exceptional) newform $f \in \mathcal{B}(N, \chi)$ or $\mathcal{B}_k(N, \chi)$ let

$$f^Z(x + iy) := \sum_n \rho_f(n) W_f(ny) G\left(\frac{n}{Z}\right) e(nx).$$

In view of (16) we can assume $y \geqslant 1/(2N)$. For notational simplicity let

$$P := t^* N.$$

First we record the trivial bound

$$(31) \qquad |f^Z(x + iy)| \ll_{A,\varepsilon} \frac{P^\varepsilon (t^* Z)^{1/2}}{N^{1/2}} \left(1 + \frac{Zy}{t^*}\right)^{-A}$$

for any $A \geqslant 0$ and any $\varepsilon > 0$. This follows directly from Cauchy-Schwarz in combination with (12) - (14) and Lemma 5 with $j = 0$. Fix $\varepsilon > 0$ and let

$$(32) \qquad Z_0 := P^\varepsilon t^*/y \leqslant 2P^\varepsilon t^* N.$$

Then we have

Lemma 7. *If $\varepsilon > 0$, then*

$$|f(x + iy)| \ll_\varepsilon P^\varepsilon \max_{Z \leqslant Z_0} |f^Z(x + iy)|,$$

and

$$(33) \qquad |f^Z(x + iy)| \ll_\varepsilon \frac{P^\varepsilon (t^* Z)^{1/2}}{N^{1/2}}.$$

In particular,

$$(34) \qquad |f(x + iy)| \ll_\varepsilon t^*(Nt^*)^\varepsilon.$$

Indeed, choosing A in (31) sufficiently large in terms of ε, it is obvious that $\sum_{Z=2^\nu \geqslant Z_0} |f^Z(x+iy)|$ is negligible, and by (10) it is enough to look at positive Fourier coefficients. This establishes the first bound, while (33) and (34) are direct consequences of (31) and (32).

We remark that Lemma 7 implies already a result of the type of Theorem 1 as soon as $Z_0 \leqslant N^{1-\delta}$, so the critical range of the dyadic decomposition is really $Z = N^{1+o(1)}$. In Proposition 1 below we will prove a version of (34) that is stronger in the t^*-aspect. We observe that we can assume

$$\text{(35)} \qquad t^* \leqslant N^{1/165},$$

for otherwise Theorem 1 follows from (34).

The aim of this section is to improve on the trivial bound if x can be "well-approximated". This is summarized in the following lemma.

Lemma 8. *Let f be a newform as above and write*

$$x = \frac{a}{q} + \beta$$

where $(a,q) = 1$, $q \leqslant N$ and $\beta \leqslant 1$, say. Let $N_2 := N/(N,q)$ as in Lemma 1. Then

$$|f^Z(x+iy)| \ll_\varepsilon P^\varepsilon (t^*)^{3/2} q \left(\beta^{3/2} Z + \frac{(t^*)^{3/2}}{Z^{1/2}} \right).$$

Proof. Using (14) and the Voronoi summation formulae (19) we find by trivial estimation that

$$\text{(36)} \qquad \begin{aligned} |f^Z(x+iy)| &= |\rho_f(1)| \left| \sum_n \lambda_f(n) e\left(n\frac{a}{q}\right) \frac{G(n/Z)W_f(ny)e(n\beta)}{\sqrt{n}} \right| \\ &\ll_\varepsilon \frac{(Nt^*)^\varepsilon}{\sqrt{N}q\sqrt{N_2}} \sum_\pm \sum_n |\lambda_{f^*}(n)| \left| \int_0^\infty \frac{G(\xi/Z)W_f(\xi y)e(\xi\beta)}{\sqrt{\xi}} \mathcal{J}^\pm \left(\frac{\sqrt{n\xi}}{q\sqrt{N_2}} \right) d\xi \right|. \end{aligned}$$

Writing $g(\xi) := G(\xi/Z)W_f(y\xi)e(\xi\beta)\xi^{-1/2}$, Lemma 5 implies

$$g^{(j)}(\xi) \ll_{j,\varepsilon} P^\varepsilon \left(\frac{t^*}{Z} \right)^{1/2} \left(\frac{t^*}{Z} + \beta \right)^j$$

for any $j \in \mathbb{N}_0$, $\varepsilon > 0$. Let us first assume

$$\text{(37)} \qquad n \geqslant X := 8P^\varepsilon q^2 N_2 \left(\beta^2 Z + \frac{(t^*)^2}{Z} \right).$$

Then $\sqrt{n\xi}/q\sqrt{N_2} \geqslant 2P^\varepsilon t^*$, and hence by (27) the integral in the second line of (36) is at most

$$\text{(38)} \qquad \ll_{\varepsilon,j} P^\varepsilon \left(\frac{t^*}{Z} \right)^{1/2} \left(\frac{q\sqrt{N_2}}{\sqrt{n}} \left(\beta\sqrt{Z} + \frac{t^*}{\sqrt{Z}} \right) \right)^j Z^{3/4} t^* \left(\frac{\sqrt{n}}{q\sqrt{N_2}} \right)^{-1/2} \ll P^{20} P^{-(j-2)\varepsilon} n^{-3/2}$$

for $1/2 \leqslant Z \leqslant Z_0$. Choosing j sufficiently large (in terms of ε), we see that these values of n contribute a negligible amount to (36). For the remaining n we use (26), and bound (36) by

$$\ll_\varepsilon \frac{P^\varepsilon}{\sqrt{N}q\sqrt{N_2}} \sum_{n \leqslant X} |\lambda_{f^*}(n)| \left(\frac{t^*}{Z} \right)^{1/2} Z^{3/4} t^* \left(\frac{\sqrt{n}}{q\sqrt{N_2}} \right)^{-1/2}.$$

We perform the sum over n by Cauchy-Schwarz and (13) obtaining

$$\ll_\varepsilon \frac{P^\varepsilon (t^*)^{3/2} Z^{1/4} X^{3/4}}{N^{1/2} q^{1/2} N_2^{1/4}}.$$

Now we use (37) together with $N_2 \leqslant N$ and Lemma 8 is immediate.

Remark. Obviously this estimate will only be useful if β and q are relatively small, i.e. x is well-approximable. For example, if $Z \approx N$, Lemma 8 is non-trivial under assumptions a little stronger than $q \leqslant N^{1/2}$ and $q\beta^{3/2} \leqslant 1/N$ which defines a subset of $[0,1]$ of measure $N^{-\delta}$.

6. Amplification

In this section we prepare the scene for a different method to estimate $|f^Z(z)| = |f^Z(x+iy)|$ that turns out to be strong if x is off rational points with small denominators. Essentially, we will consider an amplified second moment. More precisely, let f_0 be the newform whose L^∞-norm we want to bound. Let

$$(39) \qquad \max(N^{1/100}, (t^*)^3) \leqslant L \leqslant N$$

be a parameter to be chosen later, let

$$\Lambda_1 := \{p \nmid N : p \in [L, 2L] \text{ prime}\}, \quad \Lambda_2 := \{p^2 : p \in \Lambda_1\}$$

and let us define

$$(40) \qquad \alpha(\ell) := \begin{cases} \lambda_{f_0}(\ell)\bar\chi(\ell), & \ell \in \Lambda_1, \\ -\bar\chi(\ell), & \ell \in \Lambda_2, \\ 0, & \text{else.} \end{cases}$$

For $\kappa \in \{2, 4, 6, \ldots\}$ let us define

$$\mathcal{Q}_\kappa(z) := \frac{2}{\kappa-1} \sum_{f \in \mathcal{B}_\kappa(N,\chi)} \left| \sum_\ell \lambda_f(\ell)\alpha(\ell) \right|^2 |f^Z(z)|^2$$

and

$$\mathcal{Q}(z) := \sum_{\kappa \geqslant 2 \text{ even}} 2(\kappa-1)\dot\phi(\kappa)\mathcal{Q}_\kappa(z)$$

$$+ 4\pi \sum_{f \in \mathcal{B}(N,\chi)} \tilde\phi(t_f) \left| \sum_\ell \lambda_f(\ell)\alpha(\ell) \right|^2 \left| \sum_n \rho_f(n)G(n/Z)W_{f_0}(ny)e(nx) \right|^2$$

$$+ \sum_{\substack{\chi_1\chi_2=\chi \\ b \in \mathcal{B}(\chi_1,\chi_2)}} \int_\mathbb{R} \tilde\phi(t) \left| \sum_\ell \lambda_{t,\chi_1,\chi_2,b}(\ell)\alpha(\ell) \right|^2 \left| \sum_n \rho_{t,\chi_1,\chi_2,b}(n)G(n/Z)W_{f_0}(ny)e(nx) \right|^2 dt.$$

Here ϕ is the function defined in (28) with some small $B \leqslant 5$ and some very large A to be chosen later, and $\dot\phi$ and $\tilde\phi$ are the integral transforms (20) of ϕ that satisfy the relations (29) and (30). Using Lemma 7 and (29), we see that the κ-sum in $\mathcal{Q}(z)$ is absolutely convergent. The quantity $\mathcal{Q}(z)$ is not really an amplified second moment, but rather a kind of "fake moment", since we consider truncated cusp forms and we need the weight function W_{f_0} to be independent of f. Finally we remark that the amplifier that we chose is most likely not the most efficient, but it seems hard to show that the more natural amplifier $\alpha(\ell) = \lambda_{f_0}(\ell)$ for all $\ell \in [L, 2L]$ gives a nontrivial result. If we knew $\sum_{\ell \sim L} |\lambda_{f_0}(\ell)|^2 \gg L^{1-\varepsilon}$, then our analysis would no longer involve those $\ell \sim L^2$. This would greatly simplify the following sections and improve the exponent in Theorem 1 by about a factor 2.

From (11) and (12) we conclude $\lambda_f(p)^2 = \lambda_f(p^2) + \chi(p)$, hence

$$(41) \qquad \sum_\ell \lambda_{f_0}(\ell)\alpha(\ell) = \sum_{p \in \Lambda_1} 1 \gg LN^{-\varepsilon}.$$

Dropping all but one term in the definitions of $\mathcal{Q}_\kappa$ and $\mathcal{Q}(z)$ and using (29), (30) and (41), we see that

$$(42) \qquad |f_0^Z(z)|^2 \ll_\varepsilon P^\varepsilon L^{-2}t^* \mathcal{Q}_k(z)$$

if f_0 is a newform in $\mathcal{S}_k(N,\chi)$ for even $k > A - 5$, and

$$(43) \qquad |f_0^Z(z)|^2 \ll_\varepsilon P^\varepsilon L^{-2}(t^*)^{2B+2} \left(\mathcal{Q}(z) + \sum_{\kappa \geqslant A-4} \kappa|\dot\phi(\kappa)|\mathcal{Q}_\kappa(z) \right)$$

if f_0 is a newform in $\mathcal{S}_k(N,\chi)$ for even $k \leqslant A - 5$, or a newform in $\mathcal{L}_0^2(N,\chi)$. We proceed to bound $\mathcal{Q}_\kappa(z)$ and $\mathcal{Q}(z)$ from above. In view of (42) and (43), we need to get a bound of the form $L^{2-\delta}$ for some $\delta > 0$ and some sufficiently small L in terms of the level N. Opening the squares, using[3] (11), and applying the appropriate trace formula Lemma 2 or Lemma 3, we obtain

(44)

$$
\mathcal{Q}_\kappa(z) = \sum_{\ell_1,\ell_2} \bar{\alpha}(\ell_1)\alpha(\ell_2) \sum_{n,m} \frac{G(m/Z)\bar{W}_{f_0}(my)G(n/Z)W_{f_0}(ny)e((n-m)x)}{\sqrt{nm}} \sum_{d_1|(m,\ell_1)} \sum_{d_2|(n,\ell_2)} \bar{\chi}(d_1)\chi(d_2)
$$

$$
\times \left(\frac{1}{2\pi}\delta\left(\frac{m\ell_1}{d_1^2}, \frac{n\ell_2}{d_2^2}\right) + \sum_c \frac{i^{-\kappa}}{Nc} S_\chi\left(\frac{m\ell_1}{d_1^2}, \frac{n\ell_2}{d_2^2}; Nc\right)\phi\left(\frac{4\pi\sqrt{nm\ell_1\ell_2}}{d_1d_2Nc}\right) \right)
$$

$$
=: \mathcal{Q}_\kappa^\Delta(z) + \mathcal{Q}_\kappa^{\text{off-}\Delta}(z),
$$

say, where $\phi = J_{\kappa-1} = i^{\kappa-1}\phi_{\kappa-1,0}$ in the notation (28); and

$$
\mathcal{Q}(z) = \sum_{\ell_1,\ell_2} \bar{\alpha}(\ell_1)\alpha(\ell_2) \sum_{n,m} \frac{G(m/Z)\bar{W}_{f_0}(my)G(n/Z)W_{f_0}(ny)e((n-m)x)}{\sqrt{nm}} \sum_{d_1|(m,\ell_1)} \sum_{d_2|(n,\ell_2)} \bar{\chi}(d_1)\chi(d_2)
$$

$$
\times \sum_c \frac{1}{Nc} S_\chi\left(\frac{m\ell_1}{d_1^2}, \frac{n\ell_2}{d_2^2}; Nc\right)\phi\left(\frac{4\pi\sqrt{nm\ell_1\ell_2}}{d_1d_2Nc}\right)
$$

where $\phi = \phi_{A,B}$. We are assuming that κ and A are sufficiently large, so that in either case ϕ decays sufficiently rapidly at 0. The diagonal term $\mathcal{Q}_\kappa^\Delta(z)$ in (44) can be estimated trivially, and we summarize the result in the following lemma.

Lemma 9. *We have*

$$
\mathcal{Q}_\kappa^\Delta(z) \ll_\varepsilon P^\varepsilon t^* L
$$

for any $\varepsilon > 0$.

To see this, we estimate

$$
\mathcal{Q}_\kappa^\Delta(z) \leqslant \sum_{d_1,d_2} \sum_{\substack{\ell_1,\ell_2,n,m \\ m\ell_1=n\ell_2}} \frac{|\alpha(d_1\ell_1)\alpha(d_2\ell_2)G(md_1/Z)W_{f_0}(md_1y)G(nd_2/Z)W_{f_0}(nd_2y)|}{\sqrt{d_1d_2nm}}
$$

$$
\ll \sum_{d_1,d_2,\ell_1,\ell_2} \frac{|\alpha(d_1\ell_1)\alpha(d_2\ell_2)|(\ell_1,\ell_2)}{\sqrt{d_1d_2\ell_1\ell_2}} \sum_m \frac{1}{m}\left| W_{f_0}\left(m\frac{\ell_2d_1}{(\ell_1,\ell_2)}y\right)W_{f_0}\left(m\frac{\ell_1d_2}{(\ell_1,\ell_2)}y\right)\right|
$$

$$
\ll_\varepsilon \sum_{d_1,d_2,\ell_1,\ell_2} \frac{|\alpha(d_1\ell_1)\alpha(d_2\ell_2)|(\ell_1,\ell_2)}{\sqrt{d_1d_2\ell_1\ell_2}} t^* P^\varepsilon
$$

using Lemma 5. By (13) and the fact that $\alpha(\ell)$ is supported on prime powers (see (40)), we arrive at the bound of the lemma.

7. BOUNDING EXPONENTIAL SUMS

It remains to estimate the two similar quantities $\mathcal{Q}_\kappa^{\text{off-}\Delta}(z)$ and $\mathcal{Q}(z)$, an upper bound for which is given by

$$
\sum_{d_1,d_2,\ell_1,\ell_2} \frac{|\alpha(d_1\ell_1)\alpha(d_2\ell_2)|}{\sqrt{d_1d_2}} \sum_c \frac{1}{Nc}\left| \sum_{n,m} S_\chi(m\ell_1, n\ell_2; Nc)\phi\left(\frac{4\pi\sqrt{nm\ell_1\ell_2}}{Nc}\right) \right.
$$

$$
\left. \times \frac{G(d_1m/Z)\bar{W}_{f_0}(d_1my)G(d_2n/Z)W_{f_0}(d_2ny)e((d_2n-d_1m)x)}{\sqrt{nm}} \right|.
$$

---

[3]Note that this relation holds in particular for Eisenstein series

This is the most technical part of the paper and will be completed once we reach Lemma 11 below. Hecke eigenvalues have now disappeared from the picture (except from the amplifier), and we are left with bounding weighted exponential sums. We write

$$x = \frac{u}{N} + \beta$$

where $u \in \mathbb{Z}$ (not necessarily coprime to N) and $|\beta| \leqslant 1/N$. For notational simplicity let us temporarily write

$$(45) \qquad\qquad X_1 := Z/d_1, \quad X_2 := Z/d_2$$

which corresponds to the sizes of m and n respectively. By the rapid decay of ϕ near 0, we can assume that

$$(46) \qquad\qquad c \leqslant C := \frac{\sqrt{X_1 X_2 \ell_1 \ell_2}}{N} P^\varepsilon$$

for any $\varepsilon > 0$, for otherwise we can save as many powers of P as we wish; this is a similar argument as in (38). Opening the Kloosterman sum, we are left with

$$\sum_{\substack{d_1,d_2 \\ \ell_1,\ell_2}} \frac{|\alpha(d_1\ell_1)\alpha(d_2\ell_2)|}{\sqrt{d_1 d_2}} \sum_{c \leqslant C} \sum_{\substack{a\,(Nc) \\ (a,Nc)=1}} \frac{1}{Nc} \left| \sum_{n,m} e\left(m\left(\frac{\ell_1\bar{a}}{Nc} - \frac{d_1 u}{N} \right) + n\left(\frac{\ell_2 a}{Nc} + \frac{d_2 u}{N} \right) \right) \Phi_{\substack{d_1,d_2 \\ \ell_1,\ell_2}}(m,n;Nc;y,\beta) \right|$$

where

$$\Phi_{\substack{d_1,d_2 \\ \ell_1,\ell_2}}(m,n;Nc;y,\beta) := \frac{\bar{W}_{f_0}(d_1 my)W_{f_0}(d_2 ny)G(m/X_1)G(n/X_2)e((d_2 n - d_1 m)\beta)}{\sqrt{nm}}\phi\left(\frac{4\pi\sqrt{nm\ell_1\ell_2}}{Nc} \right).$$

Recall that $\phi = \phi_{A,B}$ as in (28) for some fixed A, B or $\phi = J_{\kappa-1}$ (where A and κ are very large). The general idea is now to observe that Φ oscillates only mildly, and so the n, m-sum can only be large if both a and $\bar{a}$ are in a certain small range. This will happen rarely and, as it turns out, only if u/N can be well-approximated in which case we can use the bound from Section 5. More precisely, in the range (46) and $Z \leqslant Z_0$ as in (32) we have by Lemma 5, (24) and (21)

$$(47) \qquad X_1^i X_2^j \frac{\partial^i}{\partial \xi^i} \frac{\partial^j}{\partial \eta^j} \Phi_{\substack{d_1,d_2 \\ \ell_1,\ell_2}}(\xi,\eta;Nc;y,\beta) \ll_{i,j,\varepsilon} \frac{t^*(Nc)^{1/2}P^\varepsilon}{(X_1 X_2)^{3/4}(\ell_1\ell_2)^{1/4}} \left(t^* + \frac{\sqrt{X_1 X_2 \ell_1 \ell_2}}{Nc} \right)^{i+j}$$

for any $\varepsilon > 0$ and $i, j \in \mathbb{N}_0$ (recall $|\beta| \leqslant 1/N$). The implied constant depends also on the choice of A and κ in ϕ, but we regard these as fixed. Applying Lemma 4 to the n- and m-sum we see that the double sum is negligible, unless both

$$(48) \qquad \left\| \frac{\ell_1\bar{a}}{Nc} - \frac{d_1 u}{N} \right\| \leqslant \frac{P^\varepsilon(t^* + \sqrt{X_1 X_2 \ell_1 \ell_2}/(Nc))}{X_1}, \quad \left\| \frac{\ell_2 a}{Nc} + \frac{d_2 u}{N} \right\| \leqslant \frac{P^\varepsilon(t^* + \sqrt{X_1 X_2 \ell_1 \ell_2}/(Nc))}{X_2},$$

for otherwise we can save as many powers of P as we wish by choosing j sufficiently large in Lemma 4. If (48) is satisfied, we cannot do better than estimating the n, m-sum trivially by

$$\ll \frac{t^*(Nc)^{1/2}(X_1 X_2)^{1/4}P^\varepsilon}{(\ell_1\ell_2)^{1/4}},$$

using (47) with $i = j = 0$. Now (48) implies

$$(49) \qquad\qquad \ell_1\bar{a} \equiv d_1 uc + r_1 \ (Nc), \quad -\ell_2 a \equiv d_2 uc + r_2 \ (Nc)$$

for certain integers

$$(50) \qquad |r_1| \leqslant R_1 := P^\varepsilon\left(t^* + \frac{\sqrt{X_1 X_2 \ell_1 \ell_2}}{Nc} \right)\frac{Nc}{X_1} \leqslant 2P^{2\varepsilon}t^*\left(\frac{\ell_1\ell_2 d_1}{d_2} \right)^{1/2},$$

$$|r_2| \leqslant R_2 := P^\varepsilon\left(t^* + \frac{\sqrt{X_1 X_2 \ell_1 \ell_2}}{Nc} \right)\frac{Nc}{X_2} \leqslant 2P^{2\varepsilon}t^*\left(\frac{\ell_1\ell_2 d_2}{d_1} \right)^{1/2},$$

cf. (45) and (46). If $d_1, d_2, \ell_1, \ell_2, r_1, r_2, c$ are given, a is determined modulo $Nc/(c, \ell_1, \ell_2)$, hence the number of solutions a modulo Nc to (49) is at most (c, ℓ_1, ℓ_2). Moreover, (49) implies

$$(51) \qquad (d_1 uc + r_1)(d_2 uc + r_2) + \ell_1 \ell_2 \equiv 0 \, (Nc)$$

as well as

$$(52) \qquad (\ell_1, c) \mid r_1, \quad (\ell_2, c) \mid r_2.$$

Summarizing the preceding discussion, we obtain the following preliminary bound for the off-diagonal term

$$(53) \qquad P^\varepsilon \sum_{\substack{d_1, d_2 \\ \ell_1, \ell_2}} \frac{|\alpha(d_1 \ell_1) \alpha(d_2 \ell_2)|}{\sqrt{d_1 d_2}} \sum_{\substack{c \leqslant C, |r_1| \leqslant R_1, |r_2| \leqslant R_2 \\ Nc|(d_1 uc + r_1)(d_2 uc + r_2) + \ell_1 \ell_2 \\ (\ell_1, c)|r_1, (\ell_2, c)|r_2}} \frac{(c, \ell_1, \ell_2) t^* Z^{1/2}}{(Nc)^{1/2} (d_1 d_2 \ell_1 \ell_2)^{1/4}}.$$

We will now split this expression depending on the sizes of $d_1, d_2, \ell_1, \ell_2 \in \Lambda_0 \cup \Lambda_1 \cup \Lambda_2$ where $\Lambda_0 := \{1\}$, cf. (40). For notational simplicity let us define

$$(54) \qquad C^{(j)} := \frac{P^\varepsilon Z (2L)^j}{N \sqrt{d_1 d_2}}, \quad R_1^{(j)} := 2 P^\varepsilon t^* (2L)^j \left(\frac{d_1}{d_2} \right)^{1/2}, \quad R_2^{(j)} := 2 P^\varepsilon t^* (2L)^j \left(\frac{d_2}{d_1} \right)^{1/2}.$$

Bounding the amplifier (40) by (15) and cutting the c-sum into dyadic blocks, we can estimate (53) as

$$(55)$$
$$\frac{P^\varepsilon t^* Z^{1/2}}{N^{1/2}} \sum_{\substack{i_1, i_2, j_1, j_2 \in \{0,1,2\} \\ 1 \leqslant i_1 + j_1 \leqslant 2 \\ 1 \leqslant i_2 + j_2 \leqslant 2}} \sum_{\substack{d_1 \in \Lambda_{i_1} \\ d_2 \in \Lambda_{i_2}}} \frac{L^{\theta(4 - i_1 - j_1 - i_2 - j_2)}}{(d_1 d_2)^{3/4} L^{j/2}} \max_{C \leqslant C^{(j)}} \frac{1}{C^{1/2}} \sum_{\substack{C \leqslant c < 2C, \ell_1 \in \Lambda_{j_1}, \ell_2 \in \Lambda_{j_2} \\ |r_1| \leqslant R_1^{(j)}, |r_2| \leqslant R_2^{(j)} \\ Nc|(d_1 uc + r_1)(d_2 uc + r_2) + \ell_1 \ell_2 \\ (\ell_1, c)|r_1, (\ell_2, c)|r_2}} (c, \ell_1, \ell_2)$$

where $j := (j_1 + j_2)/2$. Note that C is now independent of ℓ_1 and ℓ_2. We proceed to analyze the innermost (multiple) sum. Considering (51) modulo c, we find

$$(56) \qquad r_1 r_2 + \ell_1 \ell_2 = sc$$

for some integer s of size

$$(57) \qquad |s| \leqslant \frac{R_1^{(j_1)} R_2^{(j_2)} + (2L)^{j_1 + j_2}}{C} \leqslant \frac{5 P^{2\varepsilon} (t^*)^2 (2L)^{2j}}{C} =: S^{(j)}$$

by (50) and (54). Substituting this back into (51), we obtain

$$u^2 d_1 d_2 c + u(d_1 r_2 + d_2 r_1) + s \equiv 0 \, (N).$$

If r_1, r_2, s, c are given, $\ell_1 \ell_2$ is determined, and hence (up to allowed permutations of the prime factors) ℓ_1 and ℓ_2 are fixed. If v_p denotes the usual p-adic valuation, then (56) implies for every prime p the inequality $v_p(s) \geqslant \min(v_p(\ell_1) + v_p(\ell_2) - v_p(c), v_p(r_1) + v_p(r_2) - v_p(c))$ which together with (52) gives

$$(58) \qquad v_p(s) \geqslant \min(v_p(\ell_1) + v_p(\ell_2) - v_p(c), v_p(\ell_1), v_p(\ell_2), v_p(c)).$$

Since $c \leqslant 8 P^{2\varepsilon} t^* (\ell_1 \ell_2)^{1/2}$ by (54) and (32) and since ℓ_1, ℓ_2 are supported on prime powers, we obtain by (39) that $v_p(c) \leqslant \frac{1}{2}(v_p(\ell_1) + v_p(\ell_2))$ if ε was chosen sufficiently small. Therefore (58) and (52) imply that

$$(c, \ell_1, \ell_2) \leqslant (c, s, r_1, r_2).$$

We will see later that the case $(j_1, j_2) = (2, 2)$ in (55) requires extra work. If $(j_1, j_2) \neq (2, 2)$ we replace the innermost sum of (55) by

$$
(59) \qquad \sum_{\substack{c \asymp C, |s| \leqslant S^{(j)} \\ |r_1| \leqslant R_1^{(j)}, |r_2| \leqslant R_2^{(j)} \\ N | u^2 d_1 d_2 c + u(d_1 r_2 + d_2 r_1) + s}} (c, s, r_1, r_2) \leqslant \sum_{f \leqslant \min(C, S^{(j)}, R_1^{(j)}, R_2^{(j)})} f \sum_{\substack{c \asymp C f^{-1}, |s| \leqslant S^{(j)} f^{-1} \\ |r_1| \leqslant R_1^{(j)} f^{-1}, |r_2| \leqslant R_2^{(j)} f^{-1} \\ N | u^2 d_1 d_2 c + u(d_1 r_2 + d_2 r_1) + s}} 1.
$$

In the last step we used that $(\ell_1 \ell_2, N) = 1$ and (56) imply that (c, s, r_1, r_2) is coprime to N. If $(j_1, j_2) = (2, 2)$, we will not be able to bound (59) satisfactorily, but we recall that in this case $\ell_1 \ell_2$ is a perfect square, hence we only need to bound the quantity

$$
(60) \qquad \sum_{\substack{c \asymp C, |s| \leqslant S^{(j)} \\ |r_1| \leqslant R_1^{(j)}, |r_2| \leqslant R_2^{(j)} \\ sc - r_1 r_2 = \square \\ N | u^2 d_1 d_2 c + u(d_1 r_2 + d_2 r_1) + s}} (c, s, r_1, r_2) \leqslant \sum_{f \leqslant \min(C, S^{(j)}, R_1^{(j)}, R_2^{(j)})} f \sum_{\substack{c \asymp C f^{-1}, |s| \leqslant S^{(j)} f^{-1} \\ |r_1| \leqslant R_1^{(j)} f^{-1}, |r_2| \leqslant R_2^{(j)} f^{-1} \\ sc - r_1 r_2 = \square \\ N | u^2 d_1 d_2 c + u(d_1 r_2 + d_2 r_1) + s}} 1.
$$

We have now completed the transformation of the off-diagonal term; the exponential sums have been bounded by (55) together with (59) if $(j_1, j_2) \neq (2, 2)$ and (60) if $(j_1, j_2) = (2, 2)$. It is now clear that a diophantine problem has emerged that we will solve in the next section, at least for such $x \approx u/N$ that cannot be well-approximated.

8. Counting congruences

Lemma 10. *For arbitrary real numbers $C, S, R, \tilde{R} \geqslant 1$ and positive integers d_1, d_2, u, N let*

$$
\mathcal{A}_{d_1, d_2}(C, S, R, \tilde{R}; u, N) := \{C \leqslant c < 2C, |s| \leqslant S, |r_1| \leqslant R, |r_2| \leqslant \tilde{R} : N \mid u^2 d_1 d_2 c + u(d_1 r_2 + d_2 r_1) + s\}
$$

and

$$
\mathcal{A}^{\square}(C, S, R, \tilde{R}; u, N) := \{(c, s, r_1, r_2) \in \mathcal{A}_{1,1}(C, S, R, \tilde{R}; u, N) : sc - r_1 r_2 = \square\}.
$$

Let $0 < H \leqslant N$ be another parameter and assume

$$
(61) \qquad \left| \frac{u}{N} - \frac{a}{q} \right| \leqslant \frac{1}{qH}
$$

for some $q \leqslant H$ and $(a, q) = 1$. Then
$$
(62)
$$
$$
\#\mathcal{A}_{d_1, d_2}(C, S, R, \tilde{R}; u, N) \ll C \min(R, \tilde{R}) \left(\frac{S(d_1 \tilde{R} + d_2 R)}{N} + \frac{Sq}{N} + \frac{(d_1 \tilde{R} + d_2 R)^2}{qH} + \frac{d_1 \tilde{R} + d_2 R}{q} + 1 \right)
$$

and

$$
(63)
$$
$$
\#\mathcal{A}^{\square}(C, S, R, \tilde{R}; u, N) \ll (CSR\tilde{R}qN)^{\varepsilon}
$$
$$
\times \left(\frac{CS(\tilde{R} + R)}{N} + \frac{CSq}{N} + \frac{C(\tilde{R} + R)^2}{qH} + \frac{C(\tilde{R} + R)}{q} + C + \frac{\sqrt{SC} q \min(R, \tilde{R})}{N} \right).
$$

Proof. An upper bound for $\#\mathcal{A}_{d_1, d_2}(C, S, R, \tilde{R}; u, N)$ is the number of triplets (c, r_1, r_2) such that

$$
\left\| \frac{u}{N}(d_1 r_2 + d_2 r_1) - \frac{u^2 d_1 d_2 c}{N} \right\| \leqslant \frac{S}{N}.
$$

By (61), we can relax this condition to

$$
\left\| \frac{a}{q}(d_1 r_2 + d_2 r_1) - \frac{u^2 d_1 d_2 c}{N} \right\| \leqslant \frac{S}{N} + \frac{d_1 \tilde{R} + d_2 R}{qH}.
$$

Hence we need to count the number of $C \leqslant c < 2C, |r_1| \leqslant R, |r_2| \leqslant \tilde{R}, n \in \mathbb{Z}$ such that

$$a(d_1 r_2 + d_2 r_1) - qn \in \mathcal{I}(c) := \frac{u^2 d_1 d_2 cq}{N} + \left[-\frac{Sq}{N} - \frac{d_1 \tilde{R} + d_2 R}{H}, \frac{Sq}{N} + \frac{d_1 \tilde{R} + d_2 R}{H} \right].$$

The union of the intervals $\mathcal{I}(c)$, $C \leqslant c < 2C$, covers at most

$$\ll C \left(\frac{Sq}{N} + \frac{d_1 \tilde{R} + d_2 R}{H} + 1 \right)$$

integers. For a fixed integer $w \in \mathcal{I}(c)$ and fixed $(a, q) = 1$, the equation $a(d_1 r_2 + d_2 r_1) - qn = w$ has at most $O(\frac{d_1 \tilde{R} + d_2 R}{q} + 1)$ solutions n, and each of them determines at most $O(\min(R, \tilde{R}))$ pairs (r_1, r_2). This completes the proof of (62).

In order to prove (63), we write $r := r_1 + r_2$ and assume first $4sc \neq r^2$. The same argument as above shows that the number of triplets $C \leqslant c < 2C, |s| \leqslant S, |r| \leqslant R + \tilde{R}$ satisfying

(64)
$$N \mid u^2 c + ur + s$$

is

(65)
$$\ll C \left(\frac{Sq}{N} + \frac{\tilde{R} + R}{H} + 1 \right) \left(\frac{\tilde{R} + R}{q} + 1 \right).$$

Each of these determines essentially a quadruple (c, s, r_1, r_2): Indeed, the extra condition $sc - r_1 r_2 = \square$ gives

$$4\square - (2r_1 - r)^2 = 4sc - r^2 \neq 0,$$

and so there are at most $(CSR\tilde{R})^\varepsilon$ choices for r_1 and r_2.

Next we count quadruples (c, s, r_1, r_2) with $4sc = r^2$. Note that in this instance $s > 0$ since $c > 0$. Writing $c = mc_2^2$ and $s = ms_2^2$ where $m = (c, s)$, we have $r = \pm 2mc_2 s_2$. Since N is square-free, (64) gives

(66)
$$(N, m)(uc_2 \pm s_2) \equiv 0 \, (\mathrm{mod} \, N).$$

We fix m momentarily and use (61) as above. Hence the number of pairs $c_2 \asymp \sqrt{C/m}$, $s_2 \leqslant \sqrt{S/m}$ satisfying (66) is at most the number of c_2 satisfying

$$\left\| \frac{a}{q}(N, m)c_2 \right\| \leqslant \sqrt{\frac{S}{m} \frac{(N, m)}{N}} + \sqrt{\frac{C}{m} \frac{(N, m)}{qH}}$$

or in other words, the number of pairs (c_2, b) with $a(N, m)c_2 \equiv b \, (\mathrm{mod} \, q)$ and

$$|b| \leqslant \frac{\sqrt{S}q(N, m)}{\sqrt{m}N} + \frac{\sqrt{C}(N, m)}{\sqrt{m}H}.$$

We write this as

$$a \frac{(N, m)}{(q, N, m)} c_2 \equiv b' \left(\mathrm{mod} \, \frac{q}{(q, N, m)} \right), \qquad |b'| \leqslant \frac{\sqrt{S}q(N, m)}{\sqrt{m}N(q, N, m)} + \frac{\sqrt{C}(N, m)}{\sqrt{m}H(q, N, m)}.$$

Fixing b' determines c_2 modulo $q/(q, N, m)$. If $b' = 0$, then $q/(q, N, m)$ divides c_2, and each pair (m, c_2) determines at most one triple (c, s, r) which in turn gives at most $\min(R, \tilde{R})$ quadruples (c, s, r_1, r_2). Hence our count in this case can be bounded by

(67)
$$\ll \sum_{\substack{0 < mc_2^2 \leqslant C \\ c_2 \equiv 0 \, (\mathrm{mod} \, q/(q, m))}} \min(R, \tilde{R}) \ll \min(R, \tilde{R}) \sum_{m \leqslant C} \frac{(q, m)\sqrt{C}}{q\sqrt{m}} \ll q^\varepsilon \min(R, \tilde{R}) \frac{C}{q}.$$

If $b' \neq 0$, we find similarly that the number of the quadruples (c, s, r_1, r_2) counted in the present subcase is at most

$$\ll \sum_{m \leqslant C} \left(\frac{\sqrt{S}q(N,m)}{\sqrt{m}N(q,N,m)} + \frac{\sqrt{C}(N,m)}{\sqrt{m}H(q,N,m)} \right) \left(\sqrt{\frac{C}{m}} \left(\frac{q}{(q,N,m)} \right)^{-1} + 1 \right) \min(R, \tilde{R})$$

$$(68) \qquad \ll \sum_{m \leqslant C} \left(\frac{(N,m)}{m} \left(\frac{\sqrt{SC}}{N} + \frac{C}{Hq} \right) + \frac{(N,m)}{\sqrt{m}} \left(\frac{\sqrt{S}q}{N} + \frac{\sqrt{C}}{H} \right) \right) \min(R, \tilde{R})$$

$$\ll (NC)^\varepsilon \left(\frac{\sqrt{SC}q}{N} + \frac{C}{H} \right) \min(R, \tilde{R}).$$

Combining (65) (multiplied by $(CSR\tilde{R})^\varepsilon$), (67) and (68) and using $q \leqslant H$ completes the proof of (63).

9. Proof of Theorem 1

We are now ready to complete the analysis of the off-diagonal terms $\mathcal{Q}_\kappa^{\text{off}-\Delta}(z)$ and $\mathcal{Q}(z)$. Lemma 11 below is only a matter of careful book-keeping. We continue to assume (as we may by Dirichlet's approximation theorem) that

$$(69) \qquad x = \frac{u}{N} + O\left(\frac{1}{N}\right) = \frac{a}{q} + O\left(\frac{1}{N} + \frac{1}{qH}\right)$$

for some $(a, q) = 1$, $q \leqslant H$ and a parameter $H \leqslant N$ to be chosen later. The plan is to substitute (62) into (59) and (63) into (60) both of which will be inserted into (55), the former if $(j_1, j_2) \neq (2, 2)$, and the latter otherwise. We recall the notation (54) and (57). Thus, (59) can be bounded by

$$P^\varepsilon t^* L^j \min\left(\frac{d_1}{d_2}, \frac{d_2}{d_1} \right)^{1/2} \left(\frac{(t^*)^3 L^{3j}(d_1 d_2)^{1/2}}{N} + \frac{(t^*)^2 L^{2j} q}{N} + \frac{(t^*)^2 L^{2j} d_1 d_2 C}{qH} + \frac{t^* L^j (d_1 d_2)^{1/2} C}{q} + C \right).$$

To simplify the formula, we assume that

$$(70) \qquad\qquad\qquad\qquad t^* L^2 \leqslant H \leqslant N.$$

Recalling the convention $j = (j_1 + j_2)/2$, we get by (70) that $L^j (d_1 d_2)^{1/2} \leqslant L^2$ and therefore the previous display can be bounded by (recall $q \leqslant H$)

$$P^\varepsilon t^* L^j \min\left(\frac{d_1}{d_2}, \frac{d_2}{d_1} \right)^{1/2} \left(\frac{(t^*)^2 L^{2j} H}{N} + \frac{t^* L^j (d_1 d_2)^{1/2} C}{q} + C \right).$$

We substitute this into (55) (assuming $(j_1, j_2) \neq (2, 2)$) and obtain

$$P^\varepsilon (t^*)^2 \frac{Z^{1/2}}{N^{1/2}} \sum_{\substack{i_1, i_2, j_1, j_2 \in \{0,1,2\} \\ 1 \leqslant i_1 + j_1 \leqslant 2 \\ 1 \leqslant i_2 + j_2 \leqslant 2 \\ j_1 + j_2 < 4}} \sum_{\substack{d_1 \in \Lambda_{i_1} \\ d_2 \in \Lambda_{i_2}}} \frac{L^{\theta(4 - i_1 - j_1 - i_2 - j_2)}}{(d_1 d_2)^{3/4}} L^{(j_1 + j_2)/4} \min\left(\frac{d_1}{d_2}, \frac{d_2}{d_1} \right)^{1/2}$$

$$\times \left(\frac{(t^*)^2 L^{j_1 + j_2} H}{N} + \frac{t^* L^{3(j_1 + j_2)/4} (d_1 d_2)^{1/4} Z^{1/2}}{q N^{1/2}} + \frac{Z^{1/2} L^{(j_1 + j_2)/4}}{N^{1/2}(d_1 d_2)^{1/4}} \right).$$

At this point we see that the last term in second line produces a total contribution of L^2 if $j_1 = j_2 = 2$, but we can only win the game if we get a bound $L^{2-\delta}$ which is why we presently exclude this case. To bound the previous display efficiently requires a case by case analysis according to the possible values of i_1, i_2, j_1, j_2, which is completely straightforward. The worst case is $j_1 + j_2 = 3$, in which case $0 \leqslant i_1 + i_2 \leqslant 1$; this gives the bound

$$(71) \qquad\qquad P^\varepsilon (t^*)^2 L^\theta \left(\frac{(t^*)^2 Z^{1/2} L^{15/4} H}{N^{3/2}} + \frac{t^* L^3 Z}{qN} + \frac{Z L^{3/2}}{N} \right).$$

Next we assume $j_1 = j_2 = 2$, so $d_1 = d_2 = 1$. Using (63) and the bounds (54) and (57) for $j = 2$ as well as (70) and $q \leqslant H$, we estimate (60) by

$$P^\varepsilon \left(\frac{(t^*)^3 L^6}{N} + \frac{(t^*)^2 L^4 q}{N} + \frac{C(t^*)^2 L^4}{qH} + \frac{C t^* L^2}{q} + C \right) \ll P^\varepsilon \left(\frac{(t^*)^2 L^4 H}{N} + \frac{C t^* L^2}{q} + C \right).$$

Substituting this into (55), it is readily seen that the contribution of the term corresponding to $j_1 = j_2 = 2$ is dominated by (71). We summarize the previous computations in the following lemma.

Lemma 11. *Assume* (69) *for some H satisfying* (70). *Then*

$$|\mathcal{Q}_\kappa^{off-\Delta}(z)| + |\mathcal{Q}(z)| \ll P^\varepsilon (t^*)^2 L^\theta \left(\frac{(t^*)^2 Z^{1/2} L^{15/4} H}{N^{3/2}} + \frac{t^* L^3 Z}{qN} + \frac{Z L^{3/2}}{N} \right).$$

The scene has now been prepared to prove Theorem 1. Combining Lemma 9 and 11, we obtain

$$|\mathcal{Q}_\kappa(z)| + |\mathcal{Q}(z)| \ll P^\varepsilon (t^*)^2 L^\theta \left(\frac{(t^*)^2 Z^{1/2} L^{15/4} H}{N^{3/2}} + \frac{t^* L^3 Z}{qN} + \frac{Z L^{3/2}}{N} \right) + P^\varepsilon t^* L.$$

Choosing $B = 3$ and observing (29), we can substitute this bound into (42) and (43), and the κ-sum in (43) is absolutely convergent. This gives the final estimate

$$(72) \quad |f^Z(x+iy)| \ll P^\varepsilon (t^*)^5 L^{\theta/2} \left(\frac{t^* Z^{1/4} L^{7/8} H^{1/2}}{N^{3/4}} + \frac{(t^* L Z)^{1/2}}{(qN)^{1/2}} + \frac{Z^{1/2} L^{-1/4}}{N^{1/2}} \right) + P^\varepsilon (t^*)^{9/2} L^{-1/2},$$

still under the assumptions (69) and (70). In this case, the major arc bound in Lemma 8 gives

$$(73) \quad |f^Z(x+iy)| \ll P^\varepsilon (t^*)^{3/2} \left(\frac{qZ}{N^{3/2}} + \frac{Z}{H^{3/2}} + \frac{(t^*)^{3/2} q}{Z^{1/2}} \right).$$

It remains to optimize the parameters. We can assume $Z \geqslant N^{9/10}$, for otherwise we use (33). The critical terms are the first and third in (72), and the second in (73). In order to match these we use $\theta = 7/64$, (32) and choose

$$H := N^{313/457} (t^*)^{-1803/914} \qquad L := N^{64/457} (t^*)^{96/457}.$$

Under our assumption (35), this satisfies the requirements (70) and (39). Then by (32), the bound (72) becomes

$$(74) \quad |f^Z(x+iy)| \ll P^\varepsilon \left((t^*)^{\frac{9979}{1828}} N^{-\frac{25}{914}} + (t^*)^{\frac{11181}{1828}} N^{\frac{71}{914}} q^{-\frac{1}{2}} \right)$$

which gives

$$(75) \quad |f^Z(x+iy)| \ll P^\varepsilon \left((t^*)^{\frac{9979}{1828}} N^{-\frac{25}{914}} + (t^*)^{\frac{9979}{1828}} N^{\frac{6158}{75405}} q^{-\frac{1}{2}} \right)$$

upon using (35) for the second term. For (73), we use $Z \geqslant N^{9/10}$ and (32) to get

$$(76) \quad |f^Z(x+iy)| \ll P^\varepsilon \left((t^*)^{\frac{5}{2}} q N^{-\frac{1}{2}} + (t^*)^{\frac{9979}{1828}} N^{-\frac{25}{914}} + (t^*)^3 q N^{-\frac{9}{20}} \right).$$

We use (75) if $q \geqslant q_0$ and (76) if $q < q_0$. We have lots of freedom where to choose the cutting line, for example $q_0 = N^{1/3}$ does the job, and we arrive at

$$|f^Z(x+iy)| \ll P^\varepsilon (t^*)^{\frac{9979}{1828}} N^{-\frac{25}{914}}.$$

This completes the proof of Theorem 1.

10. A Theorem of Iwaniec-Sarnak

In this final section we modify the proof of (4) in order to show that it holds uniformly up to an N^ε factor for Hecke-Maaß cusp forms f of arbitrary square-free level N thus giving (7):

Proposition 1. *Let f be an L^2-normalized Hecke-Maaß cusp form of square-free level N and let $\varepsilon > 0$. Then*
$$\|f\|_\infty \ll_\varepsilon N^\varepsilon (t^*)^{5/12+\varepsilon}.$$

The notation and presentation is kept similar to [IS] so that the reader may easily compare and contrast both works. The starting point is the spectral expansion for an automorphic kernel. Let h be an even function that is holomorphic in $|\Im t| \leqslant \frac{1}{2} + \delta$ and satisfies $h(t) \ll (|t| + 1)^{-2-\delta}$ for some $\delta > 0$. Let k be the Harish-Chandra/Selberg transform of h. Then for $z \in \mathbb{H}$ we have

$$
K(z) := \sum_{\gamma \in \Gamma_0(N)/\{\pm 1\}} k(z, \gamma z) = \frac{h(i/2)}{\mathrm{vol}(X_0(N))} + \sum_{f \in \mathcal{B}(N,\chi)} h(t_f)|f(z)|^2
$$
$$
+ \sum_{\substack{\chi_1\chi_2=\chi \\ b\in\mathcal{B}(\chi_1,\chi_2)}} \frac{1}{4\pi} \int_{\mathbb{R}} h(t)|E_{t,\chi_1,\chi_2,b}(z)|^2 dt
$$

where $k(z, w) = k(u(z, w))$ and

$$u(z, w) := \frac{|z - w|^2}{4\Im(z)\Im(w)}.$$

For our application, we choose

$$h(t) = 4\pi^2 \frac{\cosh(\frac{\pi t}{2})\cosh(\frac{\pi T}{2})}{\cosh(\pi t) + \cosh(\pi T)}.$$

This function is nonnegative on $\mathbb{R}\cup i\mathbb{R}$ and satisfies $h(t) \geqslant 1$ for $T \leqslant t \leqslant T+1$. By [IS, Lemma 1.1], its Harish-Chandra/Selberg transform has the following properties:

$$
\text{(77)} \qquad
\begin{aligned}
|k(u)| &\leqslant 4T^{1/2}u^{-1/4}(u + 1)^{-5/4}, \quad u > 0 \\
k(u) &= T + O(1 + u^{1/2}T^2), \quad 0 \leqslant u \leqslant 1.
\end{aligned}
$$

Our goal is to extract geometric information from the spectral expansion by averaging over the actions of Hecke operators on $K(z)$. For positive integers n co-prime with the level N, we define

$$R_N(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = n \text{ and } 0 \leqslant c \equiv 0 \,(\mathrm{mod}\, N) \right\}.$$

Then

$$\text{(78)} \qquad (T_n K(z)) = \frac{1}{\sqrt{n}} \sum_{g \in R_N(n)} k(z, gz).$$

On the spectral side[4], we get

$$
\text{(79)} \qquad
\begin{aligned}
(T_n K(z)) =& \frac{h(i/2)\sigma(n)}{\mathrm{vol}(X_0(N))\sqrt{n}} + \sum_{f \in \mathcal{B}(N,\chi)} h(t_f)\lambda_f(n)|f(z)|^2 \\
&+ \sum_{\substack{\chi_1\chi_2=\chi \\ b\in\mathcal{B}(\chi_1,\chi_2)}} \frac{1}{4\pi} \int_{\mathbb{R}} h(t)\lambda_{t,\chi_1,\chi_2,b}(n)|E_{t,\chi_1,\chi_2,b}(z)|^2 dt.
\end{aligned}
$$

One should notice at this point that the spectral expansion of the automorphic kernel provides a natural connection between the L^2-norms of the Hecke-Maass cusp forms f weighted by their Hecke eigenvalues in (79) with geometric information in (78) for each $(n, N) = 1$. Bounding (78) and averaging over $(n, N) = 1$, along with implementing an amplifier, will produce a bound for any

---

[4]Note that here we are again using the multiplicative properties of the coefficients of our basis of Eisenstein series.

individual form f with $T \leqslant t_f \leqslant T+1$. By (16) we can assume $y \geqslant \sqrt{3}/(2N)$. Given $(n, N) = 1$ we wish to obtain a non-trivial bound for the right hand side of (78) by estimating the number of $g \in R_N(n)$ which satisfy $u(z, gz) < \delta$ for some $\delta \geqslant 0$. By (77) one sees that $k(u) \ll T$ for all $u \leqslant n^{-4}$. Therefore,

$$(80) \qquad \sum_{g \in R_N(n)} k(u(z, gz)) \quad \ll \quad TM(z, n, n^{-4}) + T^{1/2} \int_{n^{-4}}^{\infty} \delta^{-1/4}(1+\delta)^{-5/4} dM(\delta)$$

where

$$M(\delta) = M(z, n, \delta) = \#\{g \in R_N(n) \mid u(z, gz) < \delta\}.$$

We proceed as in [IS] by obtaining estimates on $M(\delta)$. We write $M(\delta) = M_0(\delta) + M_\star(\delta)$ where M_0 counts over the set of upper-triangular matrices

$$M_0(\delta) = M_0(z, n, \delta) = \#\{g \in R_N(n) \mid u(z, gz) < \delta, \text{ and } c = 0 < a\}$$

and $M_\star$ counts over those matrices whose lower left hand entry is strictly positive

$$M_\star(\delta) = M_\star(z, n, \delta) = \#\{g \in R_N(n) \mid u(z, gz) < \delta \text{ and } c > 0\}.$$

Estimates for $M_0(\delta)$ are straightforward and do not depend on the level N since $c = 0$. Therefore, one gets

$$(81) \qquad M_0(\delta) \ll_\varepsilon n^\varepsilon \left(1 + \sqrt{n\delta}y\right)$$

exactly as in [IS, A.10]. For the matrices $g \in R_N(n)$ with $c > 0$ we have

$$4nc^2y^2u(z, gz) = c^2|(cz+d)z - az - b|^2$$

or $|(cz+d)(cz-a) + n| = 2cy\sqrt{un}$ which is the same as,

$$|n - |cz+d|^2 + (cz+d)(2cx+d-a)| = 2cy\sqrt{un}.$$

For $u < \delta$, the imaginary part of the left hand side above gives

$$|2cx + d - a| \leqslant 2\sqrt{\delta n}.$$

Therefore, taking the real part produces

$$||cz+d| - \sqrt{n}| \leqslant 4\sqrt{\delta n}$$

so that

$$\left|\left|\frac{a+d}{2} + icy\right| - \sqrt{n}\right| \leqslant 5\sqrt{\delta n}.$$

Combining the above results and introducing new variables $A = \frac{a-d}{2}$ and $D = \frac{a+d}{2}$ yields the system of equations

$$(82) \qquad \begin{aligned} A &= cx + O(\sqrt{n\delta}), \\ D^2 + c^2y^2 &= n + O(n(\delta + \sqrt{\delta})), \\ A^2 &\equiv D^2 - n \,(\mathrm{mod}\ c), \\ N &\leqslant c \ll y^{-1}\sqrt{n(\delta+1)}, \\ c &\equiv 0 \,(\mathrm{mod}\ N). \end{aligned}$$

For fixed c and D, we write $(D^2 - n, c) = c_0 = c_1^2 c_2$ where c_1^2 is the square part of c_0 and c_2 is square-free. As remarked in [IS], there are at most $O(c^\varepsilon(1 + \frac{c_1}{c}\sqrt{n\delta}))$ choices of A satisfying (82) and then the only remaining matrix entry b is determined. Therefore, writing $c = mN$ and $W = (Ny)^{-1}\sqrt{n(\delta+1)}$, the number of solutions A, D, c of (82) is at most

$$(83) \qquad \sum_{1 \leqslant m \ll W} (mN)^\varepsilon \sum_{c_0 | mN} \left(1 + \frac{c_1}{mN}\sqrt{n\delta}\right) \sum_{\substack{D^2 + m^2(Ny)^2 = n + O(n(\sqrt{\delta}+\delta)) \\ (D^2-n, mN) = c_0}} 1.$$

When $\delta \geqslant 1$, we have $n + O(n(\sqrt{\delta} + \delta)) = O(n\delta)$ and therefore (83) above is (note $Ny \geqslant 1/2$)

$$
(84) \quad \ll (WN)^\varepsilon \left\{ \sum_{\substack{m,D \\ D^2 + m^2(Ny)^2 \ll n\delta}} 1 + \sqrt{n\delta} \sum_{1 \leqslant m \ll W} \sum_{c_0 | mN} \frac{c_1}{mN} \sum_{\substack{(D^2 - n) + m^2(Ny)^2 \ll n\delta \\ D^2 - n \equiv 0 (\bmod c_0)}} 1 \right\}
$$

$$
\ll (WN)^\varepsilon \left\{ n\delta + \sqrt{n\delta} \sum_{1 \leqslant m \ll W} \sum_{c_0 | mN} \frac{\sqrt{n\delta}}{mN} \right\} \ll n\delta(n\delta N)^\varepsilon.
$$

When $\delta < 1$, then $\sqrt{\delta}$ dominates in $O(n(\sqrt{\delta} + \delta))$ and (83) above is

$$
\ll (WN)^\varepsilon \left\{ \sum_{\substack{D,m \\ D^2 + m^2(Ny)^2 - n \ll n\sqrt{\delta}}} 1 + \sqrt{n\delta} \sum_{1 \leqslant m \ll W} \sum_{c_0 | mN} \frac{c_1}{mN} \sum_{\substack{(D^2 - n) + m^2(Ny)^2 \ll n\sqrt{\delta} \\ D^2 - n \equiv 0 (\bmod c_0)}} 1 \right\}.
$$

By [IS, Lemma 1.4] we have $\#\{r, s \in \mathbb{Z} \mid |r^2 + Ys^2 - n| \leqslant n\Delta\} \ll_\varepsilon n^\varepsilon(n\Delta^{1/2} + 1)$ for $Y \geqslant 1/2$ and $\Delta \leqslant 1$, hence we can bound the previous display by

$$
(85) \quad \ll (WN)^\varepsilon \left\{ (n\delta^{1/4} + 1) + \sqrt{n\delta} \sum_{1 \leqslant m \ll W} \sum_{c_0 | mN} \frac{c_1}{mN} \left(1 + \frac{\sqrt{n}\delta^{1/4}}{c_1} \right) \right\}
$$

$$
\ll (WN)^\varepsilon \left\{ (n\delta^{1/4} + 1) + \sqrt{n\delta} \sum_{\ell^2 m \ll W} \frac{1}{\ell} + n\delta^{3/4} \right\} \ll (nN)^\varepsilon (1 + n\delta^{1/4}).
$$

Combining the bounds (84), (85) and (81) gives

$$
M(\delta) \ll_\varepsilon (nN)^\varepsilon \left(1 + n(\delta^{1/4} + \delta^{1+\varepsilon}) + \sqrt{n\delta}y \right).
$$

In view of (80), one sees that we have arrived at the following bound for the geometric side upon integration by parts:

$$
(86) \quad \sum_{g \in R_N(n)} k(u(z, gz)) \ll_\varepsilon (nN)^\varepsilon \left(T + T^{1/2}n + T^{1/2}n^{1/2}y \right).
$$

This bound is the same as in [IS, A.11] with the exception of an additional N^ε level factor. Next we implement an amplifier.

Let $\{\alpha_\ell\}$ be arbitrary weights supported on $(\ell, N) = 1$ and $\ell \leqslant L$ for some parameter L. Let f_0 be the L^2-normalized cusp form we wish to bound, and let $t_{f_0} \sim T$. Using the multiplication law (11), we see

$$
|f_0(z)|^2 \left| \sum_{\ell \leqslant L} \alpha_\ell \lambda_{f_0}(\ell) \right|^2 \leqslant \sum_{f \in \mathcal{B}(N,\chi)} h(t_f) |f(z)|^2 \left| \sum_{\ell \leqslant L} \alpha_\ell \lambda_f(\ell) \right|^2
$$

$$
\leqslant \sum_{\ell_1, \ell_2 \leqslant L} \alpha_{\ell_1} \overline{\alpha_{\ell_2}} \sum_{d | (\ell_1, \ell_2)} \frac{\chi(d)d}{\sqrt{\ell_1\ell_2}} \sum_{g \in R_N\left(\frac{\ell_1\ell_2}{d^2}\right)} k(z, gz)
$$

where in the last inequality we added the nonnegative contribution of the constant function and the Eisenstein series. With the resulting bound (86) for the geometric side, the right hand side above is bounded by

$$
(LN)^\varepsilon \sum_{\ell_1, \ell_2 \leqslant L} |\alpha_{\ell_1} \alpha_{\ell_2}| \sum_{d | (\ell_1, \ell_2)} \frac{d}{\sqrt{\ell_1\ell_2}} \left(T + T^{1/2} \left(\frac{\ell_1\ell_2}{d^2} \right) + T^{1/2} \left(\frac{\ell_1\ell_2}{d^2} \right)^{1/2} y \right).
$$

By Cauchy-Schwarz, the first term in parentheses contributes $T(LN)^\varepsilon \sum |\alpha_\ell|^2$, the second contributes $T^{1/2}L(LN)^\varepsilon (\sum |\alpha_\ell|)^2$, while the last term contributes $T^{1/2}y(LN)^\varepsilon (\sum |\alpha_\ell|)^2$. Note that the last term contribution is actually sharper than listed in [IS] by an $L^{1/2}$ factor. Thus we get the total bound

$$|f_0(z)|^2 \left| \sum_{\ell \leqslant L} \alpha_\ell \lambda_{f_0}(\ell) \right|^2 \ll (LN)^\varepsilon \left(T \sum_{\ell \leqslant L} |\alpha_\ell|^2 + (L+y)T^{1/2} \left(\sum_{\ell \leqslant L} |\alpha_\ell| \right)^2 \right).$$

Now we set, similar to (40),

$$\alpha_\ell := \begin{cases} \lambda_{f_0}(\ell)\bar{\chi}(\ell) & \text{if } \ell = p \leqslant \sqrt{L} \\ -\bar{\chi}(\ell) & \text{if } \ell = p^2 \leqslant L \\ 0 & \text{otherwise} \end{cases}$$

for all $p \nmid N$. By (41) and (13) this yields

$$|f_0(z)|^2 L(LN)^{-\varepsilon} \ll_\varepsilon (LN)^\varepsilon (TL^{1/2} + (L+y)T^{1/2}L),$$

and choosing $L = T^{1/3}$ we get that

(87) $$f_0(z) \ll (tN)^\varepsilon (t^{5/12} + y^{1/2}t^{1/4})$$

where $t = t_{f_0} \geqslant 2$ (without loss of generality). This is uniform in N and completes the proof of Proposition 1 if $y \leqslant t^{1/3}$. If $y > t^{1/3}$, we follow the corrigendum to [IS, Lemma A.1] in [Sa1]. From the Fourier expansion, (9), (10) and (23) we have

$$|f_0(z)| \ll |\rho_{f_0}(1)| \left(\sum_{n \leqslant t/y} |\lambda_{f_0}(n)| y^{1/2} \cosh(\pi t/2) |K_{it}(2\pi ny)| + \sum_{n > t/y} \frac{|\lambda_{f_0}(n)|}{\sqrt{n}} e^{-ny} \right).$$

The tail is negligible for $y > t^{1/3}$, and we can clearly assume $y \leqslant t$. By (14), Cauchy-Schwarz and (13) we find

(88) $$|f_0(z)| \ll_\varepsilon N^{-1/2}(Nt)^\varepsilon t^{1/2} \left(\sum_{n \leqslant t/y} |\cosh(\pi t/2)K_{it}(2\pi ny)|^2 \right)^{1/2}.$$

for any $\varepsilon > 0$. A uniform asymptotic expansion for the Bessel-K function is stated in [Ba1, (2)] and proved in [Ba2]:

$$K_{it}(tx) = \frac{\pi\sqrt{2}}{t^{1/3}} e^{-\pi t/2} \left(\frac{\zeta}{x^2-1} \right)^{1/4} \left(\text{Ai}(t^{2/3}\zeta) + O\left(\frac{1}{(1+t^{2/3}|\zeta|)^{1/4}t} \right) \right), \quad t > 1, x > 0$$

where $\zeta \in \mathbb{R}$ is given by $\frac{2}{3}\zeta^{3/2} = (x^2-1)^{1/2} - \text{arcsec}(x)$ and $\text{Ai}(x) \ll (1+|x|)^{-1/4}$ is the Airy-function. Since $\zeta \sim (x-1)$ for $1/2 \leqslant x \leqslant 3/2$, we find by trivial estimates

$$\cosh(\pi t/2)K_{it}(w) \ll \min(t^{-1/3}, |w^2 - t^2|^{-1/4})$$

for $t > 1, w > 0$. Now we estimate the n-sum in (88). The transition range $2\pi ny = t + O(t^{1/3})$ contributes at most

$$\ll \left(\frac{t^{1/3}}{y} + 1 \right) t^{-2/3}$$

to the n-sum, while the contribution of the remaining n can be bounded by

$$\ll \sum_{\substack{n \leqslant t/y \\ |2\pi ny - t| > t^{1/3}}} \frac{1}{|(2\pi ny)^2 - t^2|^{1/2}}$$

$$\leqslant \int_0^{\frac{t - t^{1/3}}{2\pi y}} \frac{d\xi}{(t^2 - (2\pi y\xi)^2)^{1/2}} + O\left(\frac{1}{t^{2/3}}\right) + \int_{\frac{t + t^{1/3}}{2\pi y}}^{t/y} \frac{d\xi}{((2\pi y\xi)^2 - t^2)^{1/2}}$$

$$\ll \frac{1 + \log(ty)}{y} + \frac{1}{t^{2/3}}$$

for $t, y > 1$. Collecting the previous estimates we find

$$|f_0(z)| \ll N^{-1/2} \left(\left(\frac{t}{y}\right)^{1/2} + t^{1/6}\right)(Nty)^\varepsilon$$

which implies Proposition 1 if $t^{1/3} < y \leqslant t$.

References

[AU]   A. Abbes, E. Ullmo, *Comparison des métriques d'Arakelov et de Poincaré sur $X_0(N)$*, Duke Math. J. **80** (1995), 295-307 1, 5

[Ba1] C. B. Balogh, *Uniform asymptotic expansions of the modified Bessel function of the third kind of large imaginary order*, Bull. Amer. Math. Soc **72** (1966), 40-43 21

[Ba2] C. B. Balogh, *Asymptotic expansions of the modified Bessel function of the third kind of imaginary order*, SIAM J. Appl. Math. **15** (1967), 1315-1323 21

[DFI] W. Duke, J. Friedlander, H. Iwaniec, *The subconvexity problem for Artin L-functions*, Invent. Math. **149** (2002), 489–577 5

[Fr]   E. Freitag, *Hilbert modular forms*, Springer-Verlag, Berlin 1990 1

[GJ]   S. Gelbart, H. Jacquet, *Forms on GL_2 from the analytic point of view*, In: Automorphic forms, representations, and L-functions (A. Borel, W. Casselman eds.), Part 1, Proc. Sympos. Pure Math. **33** (1979), 213–251 4

[GR]   I. S. Gradshteyn, I. M. Ryzhik, *Tables of integrals, series, and products*, 5th edition, Academic Press, New York, 1994 8

[HM]   G. Harcos, P. Michel, *The subconvexity problem for Rankin–Selberg L-functions and equidistribution of Heegner points. II*, Invent. Math. **163** (2006), 581–655 1, 5, 7

[Iw1] H. Iwaniec, *Spectral methods of automorphic forms*, 2nd edition, Graduate Studies in Mathematics 53, American Mathematical Society, Providence, RI; Revista Matemática Iberoamericana, Madrid, 2002 2, 6

[Iw2] H. Iwaniec, *Topics in classical automorphic forms*, Graduate Studies in Mathematics 17, American Mathematical Soceity, Providence, RI, 1997 2, 6

[IS]   H. Iwaniec, P. Sarnak, *L^∞ norms of eigenfunctions of arithmetic surfaces*, Annals of Math. **141** (1995), 301-320 2, 4, 18, 19, 20, 21

[JK1] J. Jorgenson, J. Kramer, *Bounding the sup-norm of automorphic forms*, GAFA **14** (2004), 1267-1277 1

[JK2] J. Jorgenson, J. Kramer, *Bounds on Faltingss delta function through covers*, Annals of Math. **170** (2009), 1-42. 1

[KS]   H. Kim, *Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2 (with Appendix 1 by D. Ramakrishnan and Appendix 2 by H. Kim and P. Sarnak)*, J. Amer. Math. Soc. **16** (2003), 139–183 4, 5

[KMV] E. Kowalski, P. Michel and J. VanderKam, *Rankin-Selberg L-functions in the level aspect*, Duke Math. J. **114** (2002), 123-191 5

[Ku]   P. Kurlberg, *Bounds on supremum norms for Hecke eigenfunctions of quantized cat maps*, Ann. Henri Poincaré **8** (2007), no. 1, 75–89. 2

[MU]   P. Michel, E. Ullmo, *Points de petite hauteur sur les courbes modulaires $X_0(N)$*, Invent. math. **131** (1998), 645-674 1, 3

[MV]   P. Michel, A. Venkatesh, *Equidistribution, L-functions and ergodic theory: on some problems of Yu. Linnik*, Proc. Int. Congr. Math. (Madrid 2006) Vol. II, 421-457, Eur. Math. Soc., Zürich, 2006 1

[Ru]   Z. Rudnick, *On the asymptotic distribution of zeros of modular forms*, IMRN **2005**, no. 34, 2059-2074 1

[Sa]   P. Sarnak, *Spectra of hyperbolic surfaces*, Bull. Amer. Math. Soc. (New Series) **40** (2003), 441-478 3

[Sa1] P. Sarnak, *Letter to Reznikov, Letter to Morawetz*, available at http://www.math.princeton.edu/sarnak 1, 21

[Sp]   F. Spinu, *The L^4-norm of the Eisenstein series*, Princeton Ph.D. thesis 2003, available at http://www.math.jhu.edu/ fspinu/ 3

[Xia] H. Xia, *On L^∞ norms of holomorpic cusp forms*, J. Number Theory **127** (2007), 325-327 2

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, 40 ST. GEORGE STREET, TORONTO, ONTARIO, CANADA, M5S 2E4

E-mail address: `vblomer@math.toronto.edu`

E-mail address: `romanh@math.toronto.edu`